

# Theory of excitations of the condensate and non-condensate at finite temperatures

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**Summary.** — We give an overview of the current theory of collective modes in trapped atomic gases at finite temperatures, when the dynamics of the condensate and non-condensate must both be considered. A simple introduction is given to the quantum field formulation of the dynamics of an interacting Bose-condensed system, based on equations of motion for the condensate wavefunction and single-particle Green's functions for the non-condensate atoms. We discuss the nature of excitations in the mean-field collisionless region, including the Beliaev second-order approximation for the self-energies. We also sketch the derivation of coupled two-fluid hydrodynamic equations using a simple kinetic equation which includes collisions between condensate and non-condensate atoms.

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## 1. — Introduction

There are several excellent review articles on trapped Bose-condensed atomic gases written at a relatively introductory level [1, 2], with emphasis on the dynamics of the condensate at  $T = 0$ . In the present lectures, I will concentrate on the interplay between the condensate and non-condensate components. This topic requires a more sophisticated analysis based on the concepts and methods of many body theory. My lectures will attempt to give a very basic introduction to this kind of approach. The primary

audience I have in mind are graduate students and postdocs coming from atomic and laser physics, rather than condensed matter theory. However, I hope the experts will also learn something from these lectures. In his lectures, Fetter [3] has given a detailed review of excitations in a trapped dilute Bose gas at  $T = 0$ . He shows that a convenient way of discussing these excitations is to start from the time-dependent Gross-Pitaevskii (GP) equation for the macroscopic wavefunction  $\Phi(\mathbf{r}, t)$ . Linearizing around the equilibrium value  $\Phi_0(\mathbf{r})$ , one finds the macroscopic wavefunctions are given by the well-known Bogoliubov coupled equations. For a uniform Bose-condensed gas, the excitation frequencies are those first discussed by Bogoliubov in 1947 [4].

At  $T = 0$ , one can assume that all the atoms are in the Bose condensate described by  $\Phi(\mathbf{r}, t)$ . In contrast, in these lectures, my main topic will be to review our understanding of what excitations are in a Bose-condensed gas at *finite* temperatures, when there is a large number of atoms in the non-condensate (in trapped gases, the non-condensate is often referred to as the “thermal cloud”). I will make some contact with the ideas discussed in the lectures by Burnett [5].

The first half of these lectures (Sections 2 - 7) deals with excitations whose very existence depends on self-consistent mean fields (of various kinds!), rather than on the collisions between atoms. In the standard language developed in condensed matter physics in the 1960's, this means the excitations are in the “collisionless region”.

The second half of the lectures (Sections 8 - 12) deals with excitations in the collision-dominated hydrodynamic region. I review the two-fluid hydrodynamic equations such as given by Landau [6], generalized to include a trap. I give an explicit microscopic derivation of such two-fluid equations in a trapped Bose gas. This extends recent work of Zaremba, Griffin and Nikuni (ZGN) [7] to include the case when the condensate is not yet in local equilibrium with the non-condensate atoms.

The type of questions I want to address in these lectures include:

1. What is the difference between an elementary excitation and a collective mode?
2. Can we *isolate* the dynamical role of the condensate on the nature of the excitations?
3. At  $T = 0$ , with a pure condensate, excitations in a gas must be oscillations of the condensate. In contrast, above  $T_{\text{BEC}}$ , the excitations are not related to a condensate. What happens to an excitation as we go from  $T = 0$  to  $T > T_{\text{BEC}}$ . How does the excitation get rid of its “condensate” dressing?
4. What is the essential physics behind the different mean-field theories of excitations which have been discussed in the recent literature: Gross-Pitaevskii, Hartree-Fock-Popov, Hartree-Fock-Bogoliubov?
5. What is the physics behind the dreaded Beliaev second-order approximation? This is the first approximation which includes damping of the elementary excitations even at  $T = 0$  [8].

6. Why are the excitations and collective modes in a Bose-condensed system uniquely interesting, compared to all other many body systems? The key reason is, of course, that the condensate couples and hybridizes single-particle excitations with density fluctuations. Above  $T_{\text{BEC}}$ , these two excitation branches are uncoupled. Once this is understood, one sees why more detailed and systematic experimental studies of excitations in trapped atomic gases at finite temperature are of great importance.

To give a careful discussion of all these questions would need 10 lectures. In these 3 lectures, I will give a speeded-up version. I will often use a *uniform* weakly interacting Bose gas to illustrate the structure of the theory. While I will only sketch the math, I will still try to give a flavour of what is involved. The approach I will use to discuss these questions is based on the field-theoretic formulation of a Bose-condensed system of particles. As I review elsewhere in this volume [9], this powerful formalism was introduced by Beliaev [10] in 1957 and extensively developed in the Golden Period: 1958-1965. I will introduce this formalism in a very schematic manner - but even experimentalists will find it useful to know some of the “language” used in this approach. There are other methods to deal with collective modes in Bose gases but they are not as useful at *isolating* the dynamical role of the condensate, or dealing with finite temperatures.

While I will always have trapped atomic Bose gases in mind, much of the general theory [11, 12] is valid for any Bose-condensed fluid (gas or liquid). Thus I will often make references to superfluid  $^4\text{He}$ , pointing out similarities with Bose gases.

## 2. – Elementary excitations and density fluctuations in normal systems

We work with quantum field operators :

$$(1) \quad \begin{aligned} \hat{\psi}(\mathbf{r}) &= \text{destroys an atom at } \mathbf{r} \\ \hat{\psi}^+(\mathbf{r}) &= \text{creates an atom at } \mathbf{r}. \end{aligned}$$

These operators satisfy the usual Bose commutation relations, such as  $[\hat{\psi}(\mathbf{r}), \hat{\psi}^+(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}')$ . Of course, if we have several hyperfine atomic states, then we have different field operators  $\hat{\psi}_a(\mathbf{r})$ , where  $a$  is the hyperfine state label (the analogue of a spin label). All observables can be written in terms of these quantum field operators. Two important examples are the local density operator:

$$(2) \quad \hat{n}(\mathbf{r}) = \hat{\psi}^+(\mathbf{r})\hat{\psi}(\mathbf{r})$$

and the Hamiltonian

$$(3) \quad \hat{H} = \int d\mathbf{r} \hat{\psi}^+(\mathbf{r}) \left[ \frac{-\nabla^2}{2m} + U_{\text{ext}}(\mathbf{r}) - \mu \right] \hat{\psi}(\mathbf{r}) + \frac{1}{2}g \int d\mathbf{r} \hat{\psi}^+(\mathbf{r})\hat{\psi}^+(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}).$$

In these lectures, the two-particle interaction will always be approximated by a  $s$ -wave scattering length, appropriate to a dilute Bose gas at very low temperatures [5, 13]. Thus  $v(\mathbf{r} - \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}')$ , with  $g = \frac{4\pi a}{m}$  (we set  $\hbar = 1$  throughout this article).

What we want to calculate are various kinds of correlation functions involving different local operators:

$$\begin{aligned}
 \langle \hat{n}(\mathbf{r}, t) \hat{n}(\mathbf{r}', t') \rangle &\sim \chi_{nn}(\mathbf{r}t, \mathbf{r}'t') \\
 &= \chi_{nn}(1, 1') \equiv \text{density response function} \\
 \langle \hat{j}_\alpha(\mathbf{r}, t) \hat{j}_\beta(\mathbf{r}'t') \rangle &\sim \chi_{j_\alpha j_\beta}(\mathbf{r}t, \mathbf{r}'t') \\
 (4) \qquad &= \chi_{j_\alpha j_\beta}(1, 1') \equiv \text{current response function.}
 \end{aligned}$$

In these tutorial lectures, I will not bother to distinguish between the various kinds of correlation functions (time-ordered, retarded, etc) or even the difference between correlation functions and response functions. These are discussed in all the standard textbooks on many body theory [12, 14] but are only important when one is doing systematic calculations. A convenient summary of the formalism is given in a recent review article on homogeneous weakly interacting Bose gases [15].

The density response function

$$(5) \qquad \chi_{nn}(1, 1') = \langle \hat{\psi}^+(1) \hat{\psi}(1) \hat{\psi}^+(1') \hat{\psi}(1') \rangle$$

involves four quantum field operators and is an example of a two-particle Green's function  $G_2(1, 1')$ . One can measure  $\chi_{nn}(1, 1')$  by coupling a weak external field to the local density in the system

$$(6) \qquad H'(t) = \int d\mathbf{r} V_{\text{ex}}(\mathbf{r}, t) \hat{n}(\mathbf{r}) \equiv \int d1 V_{\text{ex}}(1) \hat{n}(1).$$

Linear response theory gives

$$\begin{aligned}
 \delta n(1) &\equiv \langle \hat{n}(\mathbf{r}) \rangle_t - \langle \hat{n}(\mathbf{r}) \rangle_{eq} \\
 (7) \qquad &= \int d1' \chi_{nn}(1, 1') V_{\text{ex}}(1') + \dots
 \end{aligned}$$

for the density response (see Chapters 5 and 9 of Ref.[12] and Chapter 2 of Ref.[14]). For a uniform system with  $V_{\text{ex}}(\mathbf{r}, t) = V_{q,\omega} e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}$ , we have

$$\begin{aligned}
 \chi_{nn}(1, 1') &= \chi_{nn}(1 - 1') \\
 &= \chi_{nn}(\mathbf{r} - \mathbf{r}', t - t').
 \end{aligned}$$

Fourier transforming  $(\mathbf{r} - \mathbf{r}') \rightarrow \mathbf{q}$  and  $(t - t') \rightarrow \omega$ , the linear response equation in (7) reduces to

$$(8) \qquad \delta n(\mathbf{q}, \omega) = \chi_{nn}(\mathbf{q}, \omega) V_{q,\omega}.$$

If  $\chi_{nn}(\mathbf{q}, \omega) \sim \frac{1}{\omega - E_q}$  has a pole at  $\omega = E_q$ , (for further details, see Section 3) then when  $\omega$  and  $q$  of the external potential satisfy  $\omega = E_q$ ,  $\delta n(\mathbf{q}, \omega)$  as given by (8) can be very large even though  $V_{q,\omega}$  is small. We note that

$$(9) \quad \chi_{nn}(\mathbf{q}, \omega) \sim \frac{1}{\omega - E_q} \rightarrow \chi_{nn}(\mathbf{q}, t) \sim e^{-iE_q t}$$

and thus clearly the pole at  $\omega = E_q$  is the signature of an oscillating density fluctuation.

The basic correlation function in the field-theoretic approach is given by

$$(10) \quad \langle \hat{\psi}(1) \hat{\psi}^+(1') \rangle \sim G_1(1, 1').$$

This single-particle Green's function  $G_1$  involves two quantum field operators. It describes creating an atom at  $1' = \mathbf{r}', t'$ , let it propagate through the system to  $1 = \mathbf{r}, t$  and then destroying the atom. All other higher-order correlation functions such as  $G_2(1, 1')$  can be constructed out of combinations of  $G_1(1, 1')$ . For example, the *lowest order* contributions to the density response function in (5) are

$$(11) \quad \begin{aligned} \chi_{nn}(1, 1') &= \langle \hat{\psi}^+(1) \hat{\psi}(1) \hat{\psi}^+(1') \hat{\psi}(1') \rangle \\ &\simeq \langle \hat{\psi}^+(1) \hat{\psi}(1) \rangle \langle \hat{\psi}^+(1') \hat{\psi}(1') \rangle \rightarrow \langle n(1) \rangle \langle n(1') \rangle \\ &+ \langle \hat{\psi}^+(1) \hat{\psi}(1') \rangle \langle \hat{\psi}^+(1') \hat{\psi}(1) \rangle \rightarrow G_1(1, 1') G_1(1', 1) \\ &+ \langle \hat{\psi}^+(1) \hat{\psi}^+(1') \rangle \langle \hat{\psi}(1) \hat{\psi}(1') \rangle + \dots \end{aligned}$$

As we discuss in Section 4, the terms in the last line of (11) vanish in a normal Bose system but are finite for  $T < T_{\text{BEC}}$ . The poles of  $G_1$  correspond to what are called elementary excitations (or quasiparticles). In a uniform system, we have

$$(12) \quad \begin{aligned} G_1(1, 1') &\rightarrow G_1(\mathbf{q}, \omega) \sim \frac{1}{\omega - E_q^{sp}} \\ G_1(\mathbf{q}, t) &\sim e^{-iE_q^{sp} t}; \text{ in a free gas, we have } E_q^{sp} = \frac{q^2}{2m} \equiv \epsilon_q. \end{aligned}$$

One can show [12] that these single-particle excitations determine the thermodynamic properties of interacting systems. However, it is very difficult to directly measure the spectrum of  $G_1(1, 1')$  since one needs an external field which couples to  $\hat{\psi}(1)$ , ie, an atom reservoir. Later we will see that what makes a Bose-condensed system unique is that we can easily access  $G_1(1, 1')$  as a result of the effects of the Bose condensate.

Finally we introduce the key idea of a single-particle self-energy through Dyson's equation:

$$(13) \quad G_1 = G_0 + G_0 \Sigma G_1,$$

where  $G_1$  is the interacting single-particle Green's function,  $G_0$  is the non-interacting single-particle Green's function and all effects of the two-particle interactions are contained in the self-energy function  $\Sigma$ . In a uniform system, we can Fourier transform this Dyson equation to give

$$(14) \quad G_0(q, \omega) = G_0(q, \omega) + G_0(q, \omega)\Sigma(q, \omega)G_1(q, \omega),$$

where  $G_0(q, \omega) = \frac{1}{\omega - \epsilon_q}$ . This is easily solved to give

$$(15) \quad G_1(q, \omega) = \frac{1}{\omega - [\epsilon_q + \Sigma(q, \omega)]}.$$

Thus we see that  $G_1$  may have a single-particle pole at the quasiparticle energy

$$(16) \quad E_q^{sp} = \epsilon_q + \Sigma(q, E_q^{sp}).$$

In general,  $\Sigma(q, \omega) = \Sigma_R + i\Sigma_I$ , where  $\Sigma_I$  describes the damping of the single-particle excitations.

Field-theoretic calculations [11, 12] involve a systematic (diagrammatic) procedure to calculate  $\Sigma(q, \omega)$  and from this to obtain  $G_1(q, \omega)$ . We note that the self-energy  $\Sigma$  is by definition highly non-perturbative. As an illustration, let us consider the self-energy to first order in  $g$ . This Hartree-Fock approximation is shown in Fig.1. In our  $s$ -wave approximation, the total self-energy is simply

$$(17) \quad \Sigma_{HF} = gn + gn = 2gn,$$

and therefore (15) gives

$$(18) \quad G(\mathbf{q}, \omega) = \frac{1}{\omega - [\epsilon_q + 2gn]}.$$

Hence the normal HF excitation energy has the dispersion relation

$$(19) \quad E_q^{HF} = \frac{q^2}{2m} + 2gn.$$

### 3. – Density fluctuation spectrum in the mean-field approximation

The simplest approximation for the density response function introduced in Section 2 is [11, 12]

$$(20) \quad \chi_{nn}(1, 1') \Rightarrow \chi_{nn}^0(1, 1') = G_1(1, 1')G_1(1', 1).$$

In a uniform Bose system, the Fourier transform of this gives

$$(21) \quad \chi_{nn}^0(q, \omega) \sim \int d\mathbf{k} \int d\omega_1 \int d\omega_2 \frac{[N^0(\omega_1) - N^0(\omega_2)]}{\omega_1 - \omega_2 - \omega} A(\mathbf{k}, \omega_1) A(\mathbf{k} - \mathbf{q}, \omega_2).$$

Here  $N^0(\omega) = (e^{\beta\omega} - 1)^{-1}$  is the Bose distribution function and  $A(\mathbf{k}, \omega) \equiv$  single-particle spectral density  $\sim 2\text{Im}G_1(\mathbf{k}, \omega + i0^+)$ . If we use  $A_{HF}(k, \omega) \sim 2\pi\delta(\omega - E_k^{HF})$  as given in (18), we find (21) reduces to

$$(22) \quad \chi_{nn}^0(q, \omega) \sim \int d\mathbf{k} \frac{N^0(E_k^{HF}) - N^0(E_{k-q}^{HF})}{(E_k^{HF} - E_{k-q}^{HF}) - \omega}.$$

We note that  $\chi_{nn}^0$  has a continuum of poles given by  $\omega = (E_k^{HF} - E_{k-q}^{HF})$ . It is easy to understand the physics which gives rise to this “ideal-gas” spectrum. The Fourier transform of the local density operator in (2) is given by (for a uniform system)

$$(23) \quad \hat{n}_q = \sum_k \hat{a}_k^+ \hat{a}_{k-q}$$

Clearly  $\hat{n}_q$  creates a “particle-hole” density fluctuation with the following features:

$$(24) \quad \begin{aligned} &\text{change in energy : } E_k - E_{k-q} = \omega \\ &\text{change in momentum : } \mathbf{k} - (\mathbf{k} - \mathbf{q}) = \mathbf{q}. \end{aligned}$$

The spectrum (22) describes a broad incoherent superposition of particle-hole states [14] and is not a true collective mode, such as we discuss next.

The mean field approximation (MFA) for the density response is also called other names: SCF (self-consistent field), RPA (random phase approximation), but all involve the same physics. The MFA was introduced by Bohm and Pines in the period 1951-1953, work which had a pivotal effect in our understanding collective effects in all many-particle systems. We recall the linear response expression in (7), where  $\chi_{nn}$  is the *full* density response function for interacting Bose gas. If we introduce the self-consistent Hartree mean-field:

$$(25) \quad \delta V_{\text{Hartree}}(1) = \int d1' v(1 - 1') \delta n(1') = g \delta n(1),$$

then we can approximate the linear response equation in (7) by

$$(26) \quad \delta n(1) = \int d1' \chi_{nn}^0(1, 1') [V_{\text{ex}}(1') + \delta V_{\text{Hartree}}(1')].$$

The system is assumed to respond as if the atoms propagate independently (as described by  $\chi^0$ ) but are moving in an effective field  $V_{\text{eff}}(1')$ . In a uniform system, the Fourier

transform of (26) is

$$(27) \quad \delta n(q, \omega) = \chi_{nn}^0(q, \omega)[V_{q\omega} + g\delta n(q, \omega)],$$

which gives the well-known MFA expression for  $\chi_{nn}$  [12, 14]:

$$(28) \quad \chi_{nn}(q, \omega) = \frac{\chi_{nn}^0(q, \omega)}{1 - g\chi_{nn}^0(q, \omega)}.$$

This result for  $\chi_{nn}(q, \omega)$  may have new poles given by zeros of the denominator,

$$(29) \quad 1 - g\chi_{nn}^0(q, \omega) = 0.$$

Assuming that it is distinguishable from the incoherent ideal gas density fluctuation spectrum given by (22), this pole at  $\omega = E_q^{coll}$  is called a zero sound mode (a plasmon in charged systems). This language was first introduced in 1957 by Landau in Fermi systems [14] but the concept is generally applicable in any interacting many body system. Physically, it is clear that zero sound is a “collisionless” density oscillation arising from dynamic self-consistent mean fields.

We make a few comments about such “zero sound” collective modes:

1.  $E_q^{coll}$  and  $E_q^{sp}$  (the poles of  $G_1(q, \omega)$ ) are *both* states of an interacting many-particle system. However the collective mode disappears if there are no interactions, while the single-particle excitations still exist in a non-interacting gas.
2. Because of the low density, dynamic mean fields in normal systems are too weak to allow the existence of a well-defined (weakly-damped) zero sound mode in the Bose gases of current interest (for  $T > T_{BEC}$ ). If  $g\chi_{nn}^0(q, \omega) \ll 1$ , we can then approximate  $\chi_{nn}(q, \omega) \simeq \chi_{nn}^0(q, \omega)$ . The situation is quite different for  $T < T_{BEC}$ , when a coherent mean-field due to the condensate is present.
3. One *also* expects a collective pole to appear in  $\chi_{nn}(\mathbf{q}, \omega)$  in the collision-dominated hydrodynamic region. However this sound wave pole is induced by rapid collisions producing local equilibrium, as we discuss in Sections 8 - 12. Ordinary sound is not described by the MFA discussed above, ie, it is not the result of dynamic mean fields.

#### 4. – Green’s function formulation of excitations in a Bose-condensed system

All the formalism we have been discussing for  $T > T_{BEC}$  can be extended in a natural way to Bose-condensed systems [10, 11, 12] making use of the fundamental decomposition which separates out the condensate and non-condensate parts of the quantum fields:

$$(30) \quad \hat{\psi}(\mathbf{r}) = \langle \hat{\psi}(\mathbf{r}) \rangle + \tilde{\psi}(\mathbf{r}).$$



Here the average is over a restricted ensemble [16, 17] consistent with  $\langle \hat{\psi} \rangle \neq 0$ . The most profound way of doing this is to add a symmetry-breaking field

$$(31) \quad \hat{H}_{sb}(t) = \int d\mathbf{r} [\eta(\mathbf{r}, t) \hat{\psi}^+(\mathbf{r}) + \text{H.C.}]$$

and work with  $\hat{H}_{tot} = \hat{H}_{system} + \hat{H}_{sb}$ , taking the limit  $\eta \rightarrow 0$  at the end. This gives the system a “hunting license” [16] to have finite value of  $\langle \hat{\psi} \rangle$  and one finds

$$(32) \quad \langle \hat{\psi} \rangle_{sb} = 0 \text{ for } T > T_{\text{BEC}} ; \langle \hat{\psi} \rangle_{sb} \neq 0 \text{ for } T < T_{\text{BEC}}.$$

A key point is that if  $\langle \hat{\psi} \rangle_{sb} \neq 0$ , then a direct consequence is the existence of “anomalous” or “off-diagonal” propagators [10]

$$(33) \quad \begin{aligned} \langle \tilde{\psi}(1) \tilde{\psi}(1') \rangle_{sb} &\neq 0 \\ \langle \tilde{\psi}^+(1) \tilde{\psi}^+(1') \rangle_{sb} &\neq 0. \end{aligned}$$

These describe new condensate-induced correlations between non-condensate atoms at different space-time points 1 and 1'. In a sense, these anomalous correlation functions are as important as the macroscopic wavefunction  $\Phi(1)$ . We also note that  $\tilde{m}(1) \equiv \langle \tilde{\psi}(1) \tilde{\psi}(1) \rangle_{sb}$  is the “pair” function that Burnett [5] discusses in detail, using a different formalism. From now on, we leave the symmetry-breaking label on the averages implicit.

In addition, if  $\langle \hat{\psi} \rangle \neq 0$ , we find that correlation functions involving three non-condensate field operators can be finite

$$(34) \quad \langle \tilde{\psi}(1) \tilde{\psi}(2) \tilde{\psi}(3) \rangle \neq 0.$$

In particular,  $\langle \tilde{n}(1) \tilde{\psi}(1') \rangle \neq 0$ . This describes the condensate-induced coupling of non-condensate density fluctuations  $\tilde{n} = \tilde{\psi}^+ \tilde{\psi}$  and the single-particle field fluctuations (see Section 7).

Clearly one has to work with a  $2 \times 2$  matrix single-particle propagator  $G_1$  when  $\langle \hat{\psi} \rangle \neq 0$ , namely

$$(35) \quad \begin{aligned} G_{\alpha\beta} &= \begin{pmatrix} \langle \hat{\psi}(1) \hat{\psi}^+(1') \rangle & \langle \hat{\psi}(1) \hat{\psi}(1') \rangle \\ \langle \hat{\psi}^+(1) \hat{\psi}^+(1') \rangle & \langle \hat{\psi}^+(1) \hat{\psi}(1') \rangle \end{pmatrix} \\ &= \begin{pmatrix} \Phi(1) \Phi^*(1') + \tilde{G}_{11} & \Phi(1) \Phi(1') + \tilde{G}_{12} \\ \Phi^*(1) \Phi^*(1') + \tilde{G}_{21} & \Phi^*(1) \Phi(1') + \tilde{G}_{22} \end{pmatrix}, \end{aligned}$$

where we have introduced a  $2 \times 2$  matrix Green's function for the non-condensate atoms

$$(36) \quad \tilde{G}_{\alpha\beta} \equiv \begin{pmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{G}_{21} & \tilde{G}_{22} \end{pmatrix}.$$

Beliaev [10] effectively showed (in modern matrix notation)

$$(37) \quad \tilde{G}_{\alpha\beta} = G_0\delta_{\alpha\beta} + G_0\Sigma_{\alpha\delta}\tilde{G}_{\delta\beta},$$

where we use the standard convention that repeated indices ( $\delta = 1, 2$ ) are summed over. This is the famous Dyson-Beliaev equation, involving a  $2 \times 2$  matrix self-energy  $\Sigma_{\alpha\delta}$ . Clearly all components  $\tilde{G}_{\alpha\beta}$  will share the same single-particle excitation spectrum. In a uniform system, we have

$$(38) \quad \tilde{G}_{\alpha\beta}(q, \omega) = G_0(q, \omega)\delta_{\alpha\beta} + G_0(q, \omega)\Sigma_{\alpha\delta}(q, \omega)\tilde{G}_{\delta\beta}(q, \omega),$$

which is a set of linear algebraic equations which are easy to solve for  $\tilde{G}_{11}(\sim \tilde{G}_{22})$  and  $\tilde{G}_{12}(\sim \tilde{G}_{21})$ .

We will now use this Beliaev formalism to discuss various simple approximations for the self-energies  $\Sigma_{\alpha\beta}$ . The interaction energy in (3), namely

$$(39) \quad V_{int} = \frac{1}{2}g \int d\mathbf{r} \hat{\psi}^+(\mathbf{r})\hat{\psi}^+(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}),$$

splits into various distinct contributions when we use (30) to separate out the condensate parts (see Fig. 2). At  $T = 0$ , we can ignore the  $V_3$  and  $V_4$  contributions because so few atoms are in non-condensate. This is the famous Bogoliubov approximation [4], and it is equivalent to the linearized GP theory. In Beliaev language, the Bogoliubov self-energies are shown in Fig. 3. For a uniform gas, these give [10, 12]

$$(40) \quad \begin{aligned} \tilde{G}_{11}(p, \omega) &= \frac{\omega + \epsilon_p + n_c g}{\omega^2 - [\epsilon_p^2 + 2\epsilon_p n_c g]} \\ \tilde{G}_{12}(p, \omega) &= \frac{-n_c g}{\omega^2 - [\epsilon_p^2 + 2\epsilon_p n_c g]} . \end{aligned}$$

These Bogoliubov single-particle Green's functions contain the same physics as discussed in Fetter's lectures [3]. They clearly have poles at the frequencies  $\omega = \pm E_p$ , where

$$(41) \quad E_p = (\epsilon_p^2 + 2\epsilon_p n_c g)^{1/2}.$$

At low  $p$ , the single-particle excitation is phonon-like  $E_p = c_{Bog}p$ , with the Bogoliubov phonon velocity

$$(42) \quad c_{Bog}^2 = \frac{n_c g}{m}.$$

It turns out that this simple  $T = 0$  Bogoliubov theory already exhibits most of the structure which will always arise in Bose-condensed systems. This is why it plays the role of the ‘‘H-Atom’’ in discussions of Bose-condensed fluids [11].

In this approximation, as we have stressed, the starting point assumes that all atoms are in the condensate and hence  $n_c = n$ . However, we can still use  $\tilde{G}_{11}(\mathbf{p}, \omega)$  in (40) to estimate the depletion. This makes use of the relation

$$\begin{aligned} \tilde{n} &= n - n_c \sim \tilde{G}_{11}(1, 1) \\ (43) \quad &= \int d\mathbf{q} \int d\omega N^0(\omega - \mu) \tilde{A}_{11}(\mathbf{q}, \omega), \end{aligned}$$

which is an exact formula for  $\tilde{n}$  (see Ch.4 of Ref. [11]) in terms of the so-called single-particle *spectral density* [see (21)]. The spectral density given by the Bogoliubov approximation above is

$$(44) \quad A_{Bog}(\mathbf{q}, \omega) = u_q^2 \delta(\omega - E_q) - v_q^2 \delta(\omega + E_q),$$

showing the characteristic negative energy pole at  $\omega = -E_q$ . This last feature emphasizes that creating an atom in a Bose-condensed system involves a coherent weighted combination of creating an excitation *and* destroying an excitation. Calculating the depletion, one finds

$$\begin{aligned} \tilde{n} &= \int d\mathbf{q} \int d\omega N^0(\omega) [u_q^2 \delta(\omega - E_q) - v_q^2 \delta(\omega + E_q)] \\ &= \int d\mathbf{q} \{ N^0(E_q) u_q^2 - N^0(-E_q) v_q^2 \} \\ &= \int d\mathbf{q} \{ v_q^2 + N^0(E_q) (u_q^2 + v_q^2) \} \\ (45) \quad &= \frac{8}{3} n_c \left( \frac{n_c a^3}{\pi} \right)^{1/2}, \end{aligned}$$

a result first obtained by Bogoliubov [4] and reviewed in detail by Fetter [3]. We have made use of the key identity which the Bose distribution satisfies,

$$(46) \quad N^0(-E) = -[N^0(E) + 1].$$

We see from (45) that the  $T = 0$  depletion of the condensate formally arises from the negative energy pole in  $\tilde{G}_{11}(\mathbf{q}, \omega)$ .

It is easy to evaluate the density response function  $\chi_{nn}^0(q, \omega)$  in (20) using the Bogoliubov spectral density in (44). The negative energy poles are seen to give rise to new poles at  $\omega = E_k + E_{k-q}$  which have finite weight even at  $T = 0$ . These are in addition to the “normal” particle-hole poles of  $\chi_{nn}^0$  at  $\omega = E_k - E_{k-q}$ , as given by (22).

## 5. – Beyond the Bogoliubov approximation: classification in terms of self-energies

At finite  $T$ , a lot of atoms are thermally excited out of the condensate in a dilute Bose gas. These “normal” self-energies must be included. In the Beliaev self-energy formalism,

this was first done by Popov in 1965 [18]. We have to add the *ordinary* Hartree-Fock self-energies, as shown in Fig.4. This gives the first-order Popov approximation (the excited atoms are treated as as ideal Bose gas, and thus the approximation is not self-consistent)

$$(47) \quad \begin{aligned} \Sigma_{11}^{Popov}(p, \omega) &= 2n_c g + 2\tilde{n}^0 g \\ \Sigma_{12}^{Popov}(p, \omega) &= n_c g. \end{aligned}$$

These self-energies lead to [compare with (40)]

$$(48) \quad \tilde{G}_{11}(p, \omega) = \frac{\omega + \epsilon_p + \Delta}{\omega^2 - E_p^2} ; \quad \tilde{G}_{12}(p, \omega) = \frac{-\Delta}{\omega^2 - E_p^2} ,$$

with

$$(49) \quad \Delta \equiv \mu_{\text{HP}} - 2\tilde{n}^0 g ; \quad E_p^2 \equiv \epsilon_p^2 + 2\epsilon_p \Delta,$$

where

$$(50) \quad \mu_{\text{HP}} \equiv \Sigma_{11}(p=0, \omega=0) - \Sigma_{12}(p=0, \omega=0).$$

The “chemical potential”  $\mu_{\text{HP}}$  as defined in (50) here was introduced by Hugenholtz and Pines [19] at  $T = 0$  (and generalized to  $T \neq 0$  later [17]). HP showed that if  $\tilde{G}_{\alpha\beta}(p, \omega)$  had a gapless excitation spectrum in the  $p \rightarrow 0, \omega \rightarrow 0$  limit, then the true chemical potential  $\mu$  must satisfy  $\mu = \mu_{\text{HP}}$ . In the Popov approximation, calculation gives [20]

$$(51) \quad \mu_{\text{HP}} = 2n_c g + 2\tilde{n}^0 g - n_c g = n_c g + 2\tilde{n}^0 g,$$

and thus  $\Delta = n_c g$ , where  $n_c(T)$  is the ideal gas condensate density at temperature  $T$ . The generalized GP equation determines the chemical potential, which we find convenient to denote by  $\mu_c$ . In the Popov approximation, this equation is given by

$$(52) \quad \left\{ -\frac{\nabla^2}{2m} + U_{\text{ext}} + g n_c^0 + 2g\tilde{n}^0 \right\} \Phi = \mu_c \Phi,$$

and hence in a uniform gas, we see that  $\mu_c = \mu_{\text{HP}}$ . In the Popov approximation, the single-particle Green’s functions  $\tilde{G}_{\alpha,\beta}$  in (48) have poles given by

$$(53) \quad E_p = [\epsilon_p^2 + 2\epsilon_p n_c(T)g]^{1/2}.$$

This is formally the same as the  $T = 0$  Bogoliubov excitation frequency, except that now  $n_c(T)$  is temperature-dependent. For a more detailed discussion, see Chapter 3 of the recent review by Shi and Griffin [15].

The *full* Hartree-Fock Bogoliubov (HFB) approximation involves calculating  $\Sigma_{\alpha\beta}$  self-consistently using the complete matrix  $2 \times 2$  Beliaev propagator, as shown in Fig. 5.

We note that above  $T_{\text{BEC}}$ , the HFB reduces to the usual self-consistent Hartree-Fock approximation. Moreover, the HFB is the best *single-particle* approximation for a Bose-condensed system, in a variational sense. Within this limitation, the HFB will give the best single-particle approximation for the thermodynamic properties.

However, there is one “bad” aspect of the full HFB which was noticed in the 1960’s and which has been the subject of a several recent theoretical papers on Bose-condensed gases. Namely, the HFB does not obey the Hugenholtz-Pines (HP) theorem. In a uniform gas, this means that the single-particle excitations will have an energy gap in the long wavelength limit  $q \rightarrow 0$ . It is easy to check this by calculating the two chemical potentials introduced earlier:

$$\begin{aligned} \mu_{\text{HP}} &= 2(n_c + \tilde{n})g - (n_c + \tilde{m})g \\ (54) \quad &= g(n + \tilde{n} - \tilde{m}), \text{ using } n = n_c + \tilde{n}. \end{aligned}$$

The static equation of motion for  $\Phi = \sqrt{n_c}$  using the HFB approximation gives

$$(55) \quad \left[ -\frac{\nabla^2}{2m} + U_{\text{ext}} + gn_c + 2g\tilde{n} + g\tilde{m} \right] \sqrt{n_c} = \mu_c \sqrt{n_c} \rightarrow \mu_c = g(n + \tilde{n} + \tilde{m}).$$

One “solution” of this problem is to leave out  $\tilde{m}$  (which is the source of the problem) but keep  $\tilde{n}$ . This corresponds to the self-consistent HFP(opov) approximation [20, 21, 1] which, while approximate, has several nice features. It gives the correct excitation spectrum for both  $T \rightarrow 0$  and  $T > T_{\text{BEC}}$ . Moreover, the spectrum is gapless at all temperatures.

The origin of this “problem” with the HFB is clear. The HFB keeps all self-energies which are first order in the interaction  $g$ . However, by computing these self-energy diagrams using self-consistent  $\tilde{G}_{\alpha\beta}$  propagators, one is clearly bringing in terms to all orders in  $g$ . This brings one into dangerous territory! Moreover, one easily can check that  $\tilde{m}$  must be at least of order  $g$ . Thus the  $g\tilde{m}$  contribution to  $\Sigma_{12}$  is at least of  $O(g^2)$ . This suggests that to “fix-up” the HFB as a theory of excitations, we have to include all self-energy contributions to at least second order in  $g$ .

Moreover, it is not obvious that it is consistent to calculate  $\tilde{n}$  self-consistently [21] and ignore  $\tilde{m}$ , since the lowest order interaction contributions to both quantities are found to be of the same order. Indeed, one can show for a uniform gas that keeping the lowest order asymptotic correction to the ideal gas result gives  $\tilde{n} = \tilde{n}_{\text{cr}} - \tilde{m}$ , where  $\tilde{n}_{\text{cr}} = n(T/T_{\text{BEC}})^{\frac{3}{2}}$  is the critical density of an ideal gas at temperature  $T$  [22, 15]. However, we note that an expansion around the ideal gas results is not necessarily a good approximation in trapped gases (see also the discussion of the results in (106)). At  $T = 0$ , the Popov approximation is a poor guide for corrections to the Bogoliubov approximation since a calculation analogous to (45) gives  $\tilde{m} = 3\tilde{n}$  [15].

So, finally, we come to the work of Beliaev [8] who, in 1957, evaluated the  $\Sigma_{\alpha\beta}$  self-energies keeping all contributions up to order  $g^2$ . This was originally done at  $T = 0$ . For finite temperatures  $T \sim T_{\text{BEC}}$ , Beliaev’s work has been recently extended by several

authors [18, 15, 22]. When one includes the second-order diagrams for  $\Sigma_{\alpha\beta}$ , we note that one must be careful to also treat the HFB first-order diagrams correctly to order  $g^2$  since there are many terms which cancel.

As expected, the Beliaev-type second order calculation cures all the problems of the first-order HFB approximation. We make a few comments on the results for a uniform Bose gas. The self-energies are found to be

$$(56) \quad \begin{aligned} \Sigma_{11}(q, \omega) &= 2n_c g_R + 2\tilde{n}^{(1)}g + A(q, \omega)g^2 \\ \Sigma_{12}(q, \omega) &= n_c g_R + \tilde{m}^{(1)}g + B(q, \omega)g^2, \end{aligned}$$

with the renormalized interaction defined by

$$(57) \quad g_R = g \left[ 1 + g \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{2\epsilon_k} \right]$$

and

$$(58) \quad \tilde{m}^{(1)} = -n_c g \int \frac{d\mathbf{k}}{(2\pi)^3} \left[ \frac{2N^0(E_k) + 1}{2E_k} \right].$$

Here the anomalous and normal densities,  $\tilde{m}^{(1)} = \tilde{G}_{12}(1, 1)$  and  $\tilde{n}^{(1)} = \tilde{G}_{11}(1, 1)$ , are computed using the HFP approximation for  $\tilde{G}_{\alpha\beta}$ .  $g_R$  is the second-order approximation to the  $t$ -matrix, as shown in Fig. 6. Using these results to calculate the chemical potential, we find (for  $q, \omega \rightarrow 0$ )

$$(59) \quad \begin{aligned} \mu_{\text{HP}} &= 2n_c g_R + 2\tilde{n}^{(1)}g - n_c g_R - \tilde{m}^{(1)}g + (A - B)g^2 \\ &= n_c g_R + 2\tilde{n}^{(1)}g + \tilde{m}^{(1)}g. \end{aligned}$$

This last result gives precisely the HFB result for  $\mu_c$ , and hence the HP relation is satisfied. In the last line of (59), we have used the key result that for  $q, \omega \rightarrow 0$ ,  $(A - B)g^2 = 2\tilde{m}^{(1)}g$ , as proven at all temperatures by Talbot and Griffin [23].

We note that one can rewrite the expression in (59) in the alternative form

$$(60) \quad \mu = g \left[ n_c + \tilde{n}^{(1)} + \tilde{m}_R^{(1)} \right],$$

where we have introduced a “renormalized” anomalous density

$$(61) \quad \begin{aligned} \tilde{m}_R^{(1)} &\equiv \tilde{m}^{(1)} - n_c g \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{2\epsilon_k} \\ &= -n_c g \int \frac{d\mathbf{k}}{(2\pi)^3} \left[ \frac{2N^0(E_k) + 1}{2E_k} - \frac{1}{2\epsilon_k} \right]. \end{aligned}$$

In contrast to  $\tilde{m}^{(1)}$  in (58),  $\tilde{m}_R^{(1)}$  has no ultraviolet divergence from large  $k$  contributions. Proukakis, Burnett and coworkers [5] have pointed out that the self-consistent ladder diagram approximation for the  $t$ -matrix (see Ref.[24] and Ch.4 of Ref.[15]) can be expressed in terms of  $\tilde{m}_R$ . In a uniform gas, this relation is simply  $t = g(1 + \tilde{m}_R/n_c)$ . This suggests that the effect of  $\tilde{m}$  can be partially included by working in terms of the self-consistent  $t$ -matrix [5].

## 6. – Physics of the Beliaev approximation

We have seen in Section 5 that Beliaev's second-order calculation satisfies the HP theorem. It has another feature whose significance was only realized in the early 1960's. Using the above  $T = 0$  Beliaev expression (60) for the chemical potential  $\mu$ , one finds at  $T = 0$  (a result first found by Lee and Yang using a different approach)

$$(62) \quad \mu_{\text{Bel}} = ng \left[ 1 + \frac{32}{3} \left( \frac{na^3}{\pi} \right)^{1/2} \right],$$

where we have used [see (45)]

$$(63) \quad n = n_c + n_c \frac{8}{3} \left( \frac{n_c a^3}{\pi} \right)^{1/2} \equiv n_c + \tilde{n}.$$

One also finds that the phonon pole of  $\tilde{G}_{\alpha\beta}^{\text{Bel}}$  has a velocity given by

$$(64) \quad c_{\text{Bel}}^2 = \frac{4\pi na}{m^2} \left[ 1 + 16 \left( \frac{na^3}{\pi} \right)^{1/2} \right],$$

again expressed in terms of the total density  $n$ . One may easily check that

$$(65) \quad c_{\text{Bel}}^2 = \frac{n}{m} \frac{d\mu(n)}{dn},$$

that is, the Beliaev phonon excitation at  $T = 0$  has a velocity precisely equal to the *compressional* sound velocity.

This last result raises the basic question: why does the elementary excitation (pole of  $\tilde{G}_{\alpha\beta}$ ) have a velocity corresponding to that of a density fluctuation (pole of  $\chi_{nn}$ )? Later this strange identity was proven to be correct at  $T = 0$  to *all* orders of perturbation theory by Gavoret and Nozières [25]. To be precise, at  $T = 0$ , in any Bose-condensed fluid (liquid or gas), GN showed (for  $q, \omega \rightarrow 0$ )

$$(66) \quad \tilde{G}_{\alpha\beta}(q, \omega) \sim \frac{a}{\omega^2 - c^2 q^2} ; \chi_{nn}(q, \omega) \sim \frac{b}{\omega^2 - c^2 q^2},$$

where  $c^2$  is given by the compressibility as in (65). We discuss this equivalence in Section 7.

Let us consider the full time-dependent HFB equation of motion for  $\Phi(\mathbf{r}, t) \equiv \langle \hat{\psi}(\mathbf{r}) \rangle_t$ , given by

$$(67) \quad i \frac{\partial \Phi(\mathbf{r}, t)}{\partial t} = \left[ -\frac{\nabla^2}{2m} + U_{\text{ext}}(\mathbf{r}) + g n_c(\mathbf{r}, t) + 2g \tilde{n}(\mathbf{r}, t) + g \tilde{m}(\mathbf{r}, t) \right] \Phi(\mathbf{r}, t).$$

There are various approximations to this key equation which have been used in recent discussions of trapped Bose gases:

- *GP*: ignore  $\tilde{n}$  and  $\tilde{m}$  completely. This is valid at  $T = 0$  [1, 26].
- *Static HFP*: Keep  $n_c(\mathbf{r}, t)$  but set  $\tilde{n}(\mathbf{r}, t) = \tilde{n}_0(\mathbf{r})$  and  $\tilde{m}(\mathbf{r}, t) = 0$  [21, 27].
- *Static HFB*: Keep  $n_c(\mathbf{r}, t)$  but set  $\tilde{n}(\mathbf{r}, t) = \tilde{n}_0(\mathbf{r})$  and  $\tilde{m}(\mathbf{r}, t) = \tilde{m}_0(\mathbf{r})$  [17, 20]. As we have noted, this produces an energy gap in the excitation spectrum (since  $\mu_{\text{HP}} \neq \mu_c$ ).
- *Dynamic HFB*: Treat all dynamic mean-fields due to  $n_c(\mathbf{r}, t)$ ,  $\tilde{n}(\mathbf{r}, t)$  and  $\tilde{m}(\mathbf{r}, t)$  on an equal basis in a generalized mean-field calculation of the density response function  $\chi_{nn}(1, 1')$ . One finds [28] that the poles of  $\chi_{nn}$  given by this kind of calculation ( $\chi_{nn} \sim \frac{1}{\omega^2 - E^2}$ ) are *identical* to the poles of  $\tilde{G}_{\alpha\beta}$  given by the second-order Beliaev approximation ( $\tilde{G}_{\alpha\beta} \sim \frac{1}{\omega^2 - E^2}$ ).

Recent work on excitations at  $T \neq 0$  in trapped gases is easily understood in terms of such a generalized linear response calculation of  $\chi_{nn}$ . For example, Minguzzi and Tosi [29] only keep fluctuations in  $\delta n_c$  and  $\delta \tilde{n}$  but  $\delta \tilde{m}$  is completely ignored (dynamic HFPopov); in contrast, Giorgini [30] keeps fluctuations in both  $\delta \tilde{n}$  and  $\delta \tilde{m}$  but only those induced by the condensate fluctuations  $\delta n_c$ .

Let us briefly discuss the excitation frequencies in trapped gases at finite temperatures [5, 31]. For  $T \leq 0.4 T_{\text{BEC}}$ , GP theory ( $T = 0$ ) predictions [26] are in excellent agreement with the JILA [32] and MIT [33] experiments. For  $T \geq 0.6 T_{\text{BEC}}$ , however, there are significant differences between most current theoretical results and the measured excitation energies [21, 27]. The problem clearly is that mean-field theories must include the collective dynamics of *both* the condensate and the non-condensate. The Beliaev treatment does this, as do the papers [29, 30] mentioned above, as well as work based on a collisionless kinetic equation [7, 34]. These improved treatments can correctly obtain the in-phase rigid oscillation of the condensate and non-condensate equilibrium profiles (the so-called Kohn mode), as discussed in Section 11.

## 7. – Why are excitations in a Bose fluid so interesting?

We have seen in Section 6 that at  $T = 0$ , the elementary excitations have the same spectrum as density fluctuations. In fact, one can prove this equivalence to be a direct



consequence of the Bose broken symmetry,  $\Phi(\mathbf{r}) \equiv \langle \hat{\psi} \rangle \neq 0$ . Thus, this equivalence is valid at all  $T \leq T_{\text{BEC}}$ . Of course, as soon as  $T \neq 0$ , the phonon velocity is no longer simply given by the compressional sound velocity (65) as found at  $T = 0$ . This fundamental equivalence was only proven and understood in the early 1960's (Gavoret and Nozières [25], Hohenberg and Martin [17], Bogoliubov [16]). It was implicitly assumed by Landau [6] in his 1941 paper and later by Feynman [35] in his 1953–54 papers on excitations. Of course, the clear distinction between elementary excitations and density fluctuations in many-body systems was only clarified in the period starting from 1957. Only then were Bose fluids realized to be very special [16, 17, 25]. Today, we understand this key hidden assumption of the work of Landau and of Feynman has its microscopic basis in  $\Phi(\mathbf{r}) \neq 0$ .

Let us try to understand the origin of this equivalence of the field fluctuation and density fluctuation spectrum. The density operator in a Bose-condensed system can be decomposed using (30) as follows:

$$(68) \quad \begin{aligned} \hat{n}(\mathbf{r}) &= \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r}) \\ &= |\Phi_0(\mathbf{r})|^2 + \Phi_0(\mathbf{r})\tilde{\psi}^\dagger(\mathbf{r}) + \Phi_0^*(\mathbf{r})\tilde{\psi}(\mathbf{r}) + \tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r}), \end{aligned}$$

where the first three terms correspond to  $\hat{n}_c(\mathbf{r}) = n_{c0}(\mathbf{r}) + \delta n_c(\mathbf{r})$ . Clearly density fluctuations in a Bose-condensed system (ie,  $\Phi_0 \neq 0$ ) have a direct coupling to the single-particle field fluctuations, due to the possibility of atoms coming in or out of condensate reservoir. In momentum space, we recall that (68) is equivalent to

$$(69) \quad \hat{n}_q = \sum_k \hat{a}_k^\dagger \hat{a}_{k-q} = |\hat{a}_0|^2 + (\hat{a}_0^\dagger \hat{a}_{-q} + \hat{a}_0 \hat{a}_q^\dagger) + \tilde{n}_q.$$

Making the same decomposition as in (68) for the density response function (5) gives

$$(70) \quad \begin{aligned} \chi_{nn}(1, 1') &= \langle \hat{n}(1)\hat{n}(1') \rangle \\ &= \langle \hat{n}(1) \rangle \langle \hat{n}(1') \rangle + \langle \delta n_c(1)\delta n_c(1') \rangle \\ &\quad + \langle \delta n_c(1)\delta \tilde{n}(1') \rangle + \langle \delta \tilde{n}(1)\delta n_c(1') \rangle \\ &\quad + \langle \delta \tilde{n}(1)\delta \tilde{n}(1') \rangle. \end{aligned}$$

We note that in the Bogoliubov approximation, we have  $\delta \tilde{n} = 0$  and the equivalence of  $\chi_{nn}$  and  $\tilde{G}_{\alpha\beta}^B$  spectrum is then trivial (see also [3])

$$(71) \quad \chi_{nn}(q, \omega) \Rightarrow \chi_{nn}^B(q, \omega) \equiv n_{c0} \sum_{\alpha, \beta} \tilde{G}_{\alpha\beta}^B(q, \omega).$$

To illustrate how the hybridization of single-particle and density fluctuation arises, we discuss the time-dependent HFP approximation on which Ref. [29] is based. For simplicity, we consider the uniform gas case. In the MFA of Section 3, we have

$$(72) \quad \delta \tilde{n} = \chi^0 [V_{\text{ext}} + 2g(\delta n_c + \delta \tilde{n})],$$

and hence for  $V_{\text{ext}} \rightarrow 0$ ,

$$(73) \quad \delta\tilde{n} = \frac{\chi^0 2g\delta n_c}{1 - 2g\chi^0}.$$

Using this result in the HFP equation of motion (67), ie, with  $\delta\tilde{m} = 0$ , we find

$$(74) \quad (2g\delta\tilde{n} + g\delta n_c) = g \left( \frac{1 + 2g\chi^0}{1 - 2g\chi^0} \right) \delta n_c \equiv g' \delta n_c.$$

In this approximation, which only keeps the  $\delta\tilde{n}$  fluctuations induced by the condensate, the equation of motion for  $\delta\Phi(\mathbf{r}, t)$  reduces to the GP equation, but with  $g \rightarrow g'$ . Thus the characteristic poles will be determined by the zeroes of

$$(75) \quad [\omega^2 - (\epsilon_p^2 + 2g'n_c\epsilon_p)] \propto [\omega^2 - (\epsilon_p^2 + 2gn_c\epsilon_p)][1 - 2g\chi^0] - 4g^2 2n_c\epsilon_p\chi^0.$$

This is the same denominator (up to a common factor) exhibited by the response functions obtained by Minguzzi and Tosi [29]. This calculation illustrates how the Bogoliubov single-particle mode and the zero sound density fluctuation are coupled and hybridized.

More generally, one can prove that (at all  $T$ ):

$$(76) \quad \chi_{nn}(1, 1') = \sum_{\alpha, \beta} \int d2 \int d3 \Lambda_\alpha(1, 2) \tilde{G}_{\alpha\beta}(2, 3) \Lambda_\beta(3, 1') + \tilde{\chi}_{nn}(1, 1'),$$

where  $\Lambda_\alpha(1, 1')$  is a Bose broken-symmetry vertex function which vanishes if  $n_{c0} = 0$ . Using the “dielectric formalism” developed in the early 1970’s (for a review, see Ch.5 of Ref. [11]), one can prove that the self-energies  $\Sigma_{\alpha\beta}$  are such that  $\chi_{nn}$  and  $\tilde{G}_{\alpha\beta}$  have the same poles—and that there are none specifically associated with  $\tilde{\chi}_{nn}$ . The advantage of this diagrammatic formalism [36, 37, 38, 39, 40]

is that one manifestly sees that the poles of  $\tilde{G}_{\alpha\beta}$  and  $\chi_{nn}$  are identical, within a given approximation for the “building blocks” of the dielectric formalism. We now see why we can study elementary excitations from the density fluctuation spectrum when  $\Phi_0 \neq 0$ . To directly measure  $\tilde{G}_1(1, 1')$ , we need an external perturbation which can change the number of particles in the system. BEC indirectly provides us with such a probe!

This equivalence of the single-particle excitations with density fluctuations lies at the heart of the phenomenon of superfluidity. It essentially *restricts* the possible excited states to density fluctuations and thus ensures the stability of superfluid motion. However, as Nozières says in a very lucid review article [41], “that the  $T = 0$  superfluid equations merge with those of ordinary hydrodynamics does not alter the fact that real understanding must be based on a microscopic description based on long-range order induced by a condensate”.

## 8. – Hydrodynamic oscillations in a trapped Bose gas

In the previous sections, we have been discussing collective oscillations of the condensate and non-condensate in the mean-field or *collisionless* region. We now turn to the collision-dominated *hydrodynamic* region  $l \ll \lambda$ , where  $\lambda$  is the wavelength of the collective mode and  $l$  is the collisional mean-free-path of the elementary excitations [14]. Experiments in trapped Bose gases are just now starting to probe the hydrodynamic two-fluid region but this region is very rich in new physics and deserves careful study.

It is unfortunate that the collisionless region at  $T = 0$  described by the Gross-Pitaevskii equation is also commonly referred to as “hydrodynamic” theory [1, 2]. I guess we are stuck with this terminology, although the terms “quantum hydrodynamics” or “superfluid hydrodynamics” would be useful compromises.

We will limit our analysis to *finite* temperatures  $T \sim T_{\text{BEC}}$ , where the number of atoms in the non-condensate is comparable to the condensate ( $\tilde{N} \sim N_c$ ). We will concentrate on the *derivation* of two-fluid hydrodynamic equations starting from a simple microscopic model and try to clarify the physics. These two-fluid hydrodynamic equations are expressed in terms of fluctuations of the two components:  $n_c(\mathbf{r}, t)$ ,  $\mathbf{v}_s(\mathbf{r}, t)$  and  $\tilde{n}(\mathbf{r}, t)$ ,  $\mathbf{v}_n(\mathbf{r}, t)$ . We can use them to “derive” the two-fluid equations in the conventional Landau form, which are written in terms of the fluctuations of local thermodynamic variables (temperature, entropy, pressure, etc). We mainly consider the linearized version of the ZGN-type two-fluid equations, which give the hydrodynamic normal modes of the coupled system of two components. As a concrete example, we discuss the *in-phase* rigid motion of the equilibrium density profiles of the condensate and non-condensate.

We need to find equations of motion for both the condensate and the non-condensate. As a first approximation [7], we can use the time-dependent HF Popov equation of motion (see Section 6), for the order parameter  $\Phi(\mathbf{r}, t)$  :

$$(77) \quad i \frac{\partial \Phi(\mathbf{r}, t)}{\partial t} = \left[ -\frac{\nabla^2}{2m} + U_{\text{ext}}(\mathbf{r}) + g n_c(\mathbf{r}, t) + 2g \tilde{n}(\mathbf{r}, t) \right] \Phi(\mathbf{r}, t),$$

where  $n_c(\mathbf{r}, t) = |\Phi(\mathbf{r}, t)|^2$  and  $\tilde{n}(\mathbf{r}, t)$  is the local non-condensate density. It is clear that this equation of motion for  $n_c(\mathbf{r}, t)$  is not closed since it involves

$$(78) \quad \tilde{n}(\mathbf{r}, t) = \tilde{n}_0(\mathbf{r}) + \delta n(\mathbf{r}, t).$$

If we treat the non-condensate in (77) *statically*, this means that the condensate moves in the *static* Hartree-Fock mean field of non-condensate given by  $2g\tilde{n}_0(\mathbf{r})$  [21]. More generally, we need to have an equation of motion for the fluctuations in  $\delta\tilde{n}(\mathbf{r}, t)$  of the non-condensate (ie, the density of the excited atoms), which will be discussed in Section 9.

We can rewrite equation of motion in (77) in the *quantum hydrodynamic* variables for phase and amplitude, defined by

$$(79) \quad \Phi(\mathbf{r}, t) \equiv \sqrt{n_c(\mathbf{r}, t)} e^{i\theta(\mathbf{r}, t)},$$

where the superfluid velocity is given by

$$(80) \quad \mathbf{v}_s(\mathbf{r}, t) \equiv \frac{\nabla \theta(\mathbf{r}, t)}{m}.$$

Using (79), (77) can be shown to be equivalent to (all quantities depend on  $\mathbf{r}$  and  $t$ )

$$(81) \quad \begin{aligned} \frac{\partial n_c}{\partial t} + \nabla \cdot (n_c \mathbf{v}_s) &= 0 \\ m \left[ \frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s \right] &= -\nabla \mu_c, \end{aligned}$$

where the *superfluid chemical potential* is given by (in the dynamic HFP approximation)

$$(82) \quad \mu_c(\mathbf{r}, t) \equiv -\frac{\nabla^2 \sqrt{n_c}}{2m\sqrt{n_c}} + U_{\text{ext}}(\mathbf{r}) + gn_c(\mathbf{r}, t) + 2g\tilde{n}(\mathbf{r}, t).$$

The first term in (82) is the “quantum pressure” term, which is *ignored* in the so-called Thomas-Fermi (TF) approximation to the condensate equation of motion [1, 2]. If we omit the non-condensate term, equations (81) and (82) describe the well-known Gross-Pitaevskii approximation [1].

## 9. – Dynamics of the non-condensate atoms

Following Ref.[7], we base our discussion on the simplest possible *kinetic equation* for the distribution function  $f(\mathbf{r}, \mathbf{p}, t)$  valid in the semi-classical limit:

$$(83) \quad \begin{aligned} k_B T &\gg \hbar \omega_0 \quad (\omega_0 \equiv \text{trap frequency}) \\ k_B T &\gg gn. \end{aligned}$$

This distribution function is determined by the quantum Boltzmann equation

$$(84) \quad \left[ \frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_r - \nabla U(\mathbf{r}, t) \cdot \nabla_p \right] f(\mathbf{r}, \mathbf{p}, t) = \frac{\partial f(\mathbf{r}, \mathbf{p}, t)}{\partial t} \Big|_{\text{collision}},$$

where the effective potential

$$(85) \quad U(\mathbf{r}, t) = U_{\text{ext}}(\mathbf{r}) + 2g[n_c(\mathbf{r}, t) + \tilde{n}(\mathbf{r}, t)].$$

includes the dynamic HF field. Once we solve (84) for  $f(\mathbf{r}, \mathbf{p}, t)$ , we can find the *non-condensate* density from

$$(86) \quad \tilde{n}(\mathbf{r}, t) = \int \frac{d\mathbf{p}}{(2\pi)^3} f(\mathbf{r}, \mathbf{p}, t).$$

In our simple model, the excited atoms are still particle-like, ie,

$$(87) \quad E_p(\mathbf{r}, t) = \frac{p^2}{2m} + U(\mathbf{r}, t).$$

This last fact leads to many simplifications but restricts the results to finite temperatures close to  $T_{\text{BEC}}$ .

We now give the explicit form for the collision term in (84):

$$(88) \quad \left. \frac{\partial f}{\partial t} \right|_{\text{collision}} = C_{22}[f] + C_{12}[f].$$

The term  $C_{22}[f]$  describes collisions between excited atoms and is given by [42],

$$(89) \quad \begin{aligned} C_{22}[f] = & 2g^2 \int \frac{d\mathbf{p}_1}{(2\pi)^3} \frac{d\mathbf{p}_2}{(2\pi)^3} \frac{d\mathbf{p}_3}{(2\pi)^3} (2\pi)^3 \delta(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\ & \times 2\pi \delta(E + E_1 - E_2 - E_3) [(1+f)(1+f_1)f_2f_3 - ff_1(1+f_2)(1+f_3)]. \end{aligned}$$

In contrast,  $C_{12}[f]$  describes collisions involving one condensate atom. This kind of collision brings about equilibrium between the *excited* atoms and *condensate* atoms and it is given by (these collision terms were ignored in Ref.[7])

$$(90) \quad \begin{aligned} C_{12}[f] = & 2g^2 \int \frac{d\mathbf{p}_1}{(2\pi)^3} \frac{d\mathbf{p}_2}{(2\pi)^3} \frac{d\mathbf{p}_3}{(2\pi)^3} (2\pi)^3 \delta(\mathbf{p}_1 + \mathbf{p}_s - \mathbf{p}_2 - \mathbf{p}_3) \\ & \times 2\pi \delta(E + \epsilon_c - E_2 - E_3) (2\pi)^3 [\delta(\mathbf{p} - \mathbf{p}_1) - \delta(\mathbf{p} - \mathbf{p}_2) - \delta(\mathbf{p} - \mathbf{p}_3)] \\ & \times [n_c(1+f_1)f_2f_3 - n_cf_1(1+f_2)(1+f_3)], \end{aligned}$$

with  $E_i(\mathbf{r}, t) = \frac{p_i^2}{2m} + U(\mathbf{r}, t)$  and  $f_i \equiv f(\mathbf{r}, \mathbf{p}_i, t)$ . These two collision terms were first derived by Kirkpatrick and Dorfman [43], who considered a uniform gas and worked in a frame in which  $\mathbf{v}_s(\mathbf{r}, t) = 0$ . Our present calculation [44] is for trapped Bose gas and works in the lab frame. As a result, (90) takes into account that a condensate atom has energy  $\epsilon_c \equiv \mu_c + \frac{1}{2}mv_s^2$  and momentum  $\mathbf{p}_s = m\mathbf{v}_s$ , where  $\mu_c$  is defined in (82). We note that:

- $C_{22}[f]$  conserves number, momentum and energy of the colliding excited atoms and therefore

$$(91) \quad \begin{aligned} \int d\mathbf{p} C_{22}[f] &= 0 \\ \int d\mathbf{p} \mathbf{p} C_{22}[f] &= 0 \\ \int d\mathbf{p} E_p C_{22}[f] &= 0 \end{aligned}$$

- $C_{12}[f]$  conserves momentum and energy of the colliding atoms, with the condensate atoms having energy  $\epsilon_c$  and momentum  $m\mathbf{v}_s$ . Therefore, we have

$$(92) \quad \begin{aligned} \int d\mathbf{p}(\mathbf{p} - m\mathbf{v}_s)C_{12}[f] &= 0 \\ \int d\mathbf{p}(E_p - \epsilon_c)C_{12}[f] &= 0, \end{aligned}$$

but

$$(93) \quad \int d\mathbf{p}C_{12}[f] \equiv \Gamma_{12}[f] \neq 0.$$

The fact that  $\Gamma_{12}$  is finite follows since  $C_{12}$  does *not* conserve the number of *excited* atoms;  $\Gamma_{12}$  will be referred to as a source term.

As usual in dealing with the collision-dominated hydrodynamic region, we will assume that the collisions  $C_{22}[f]$  produce *local thermal equilibrium* among the excited atoms. This is described by the *local* Bose distribution:

$$(94) \quad f_0(\mathbf{r}, \mathbf{p}, t) = \frac{1}{e^{\beta[\frac{(\mathbf{p}-m\mathbf{v}_n)^2}{2m} + U(\mathbf{r}, t) - \tilde{\mu}]} - 1},$$

where now  $\beta, \mathbf{v}_n, U$  and  $\tilde{\mu}$  all depend on  $\mathbf{r}, t$ . One can easily check that  $C_{22}[f_0]$  vanishes exactly. This uses the key identity for Bose distribution given by (46). In contrast, one finds that substituting  $f = f_0$  in (90) gives

$$(95) \quad \begin{aligned} C_{12}[f_0] &= 2g^2 \int \frac{d\mathbf{p}_1}{(2\pi)^3} \frac{d\mathbf{p}_2}{(2\pi)^3} \frac{d\mathbf{p}_3}{(2\pi)^3} (2\pi)^3 \delta(\mathbf{p}_1 + \mathbf{p}_s - \mathbf{p}_2 - \mathbf{p}_3) \\ &\quad \times 2\pi \delta(E_1 + \epsilon_c - E_2 - E_3) (2\pi)^3 [\delta(\mathbf{p} - \mathbf{p}_1) - \delta(\mathbf{p} - \mathbf{p}_2) - \delta(\mathbf{p} - \mathbf{p}_3)] \\ &\quad \times \left[ n_c - n_c e^{-\beta(\tilde{\mu}' - \mu_c)} \right] (1 + f_{01}) f_{02} f_{03}, \end{aligned}$$

where  $\tilde{\mu}' \equiv \tilde{\mu} - \frac{1}{2}m(\mathbf{v}_n - \mathbf{v}_s)^2 \simeq \tilde{\mu}$  in a *linearized* theory. We note that  $C_{12}[f_0]$  in (95) vanishes when

$$(96) \quad \mu_c = \tilde{\mu},$$

ie, only if the condensate atoms are in *diffusive equilibrium* with the non-condensate atoms. The same kind of factor arises in the theory of the growth dynamics of a condensate in trapped Bose gases developed by Gardiner, Zoller and coworkers [45, 46].

We next derive *hydrodynamic equations* for the non-condensate atoms by taking moments of the kinetic equation:

$$(97) \quad \left[ \frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_r - \nabla U(\mathbf{r}, t) \cdot \nabla_p \right] f(\mathbf{r}, \mathbf{p}, t) = C_{22}[f] + C_{12}[f].$$

We obtain [7, 44]

$$\int d\mathbf{p}[\text{kinetic eq.}] \rightarrow \frac{\partial \tilde{n}}{\partial t} + \nabla \cdot \tilde{n}\mathbf{v}_n = \Gamma_{12}$$

$$\begin{aligned}
\int d\mathbf{p} \mathbf{p} [\text{kinetic eq.}] &\rightarrow m\tilde{n} \left( \frac{\partial}{\partial t} + \mathbf{v}_n \cdot \nabla \right) \mathbf{v}_n = -\nabla \tilde{P} - \tilde{n} \nabla U - m(\mathbf{v}_n - \mathbf{v}_s) \Gamma_{12} \\
\int d\mathbf{p} p^2 [\text{kinetic eq.}] &\rightarrow \frac{\partial \tilde{P}}{\partial t} + \nabla \cdot (\tilde{P} \mathbf{v}_n) = -\frac{2}{3} \tilde{P} \nabla \cdot \mathbf{v}_n \\
(98) \quad &+ \frac{2}{3} [\mu_c + \frac{1}{2} m(\mathbf{v}_n - \mathbf{v}_s)^2 - U] \Gamma_{12}.
\end{aligned}$$

Here  $\Gamma_{12}$  is defined in (93) and

$$\begin{aligned}
\tilde{n}(\mathbf{r}, t) &\equiv \int \frac{d\mathbf{p}}{(2\pi)^3} f_0(\mathbf{r}, \mathbf{p}, t) = \frac{1}{\Lambda^3} g_{3/2}(z) \\
(99) \quad \tilde{P}(\mathbf{r}, t) &\equiv \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{p^2}{3m} f_0(\mathbf{r}, \mathbf{p}, t)|_{v_n=0} = \frac{1}{\beta} \frac{1}{\Lambda^3} g_{5/2}(z),
\end{aligned}$$

with the local fugacity  $z(\mathbf{r}, t) = e^{\beta(\tilde{\mu}-U)}$ .

## 10. – The two-fluid hydrodynamic equations

In Sections 8 and 9, we derived a set of 7 equations for non-condensate and the 4 equations for the condensate. This gives a coupled set of equations for the variables:

$$(100) \quad n_c, \tilde{n}, \mathbf{v}_n, \mathbf{v}_s, \tilde{P} \text{ and } \Gamma_{12} \equiv \int \frac{d\mathbf{p}}{(2\pi)^3} C_{12}[f_0].$$

From (95), one sees  $\Gamma_{12}$  is function of  $f_0$ , which in turn depends on  $\beta, \mathbf{v}_n, \tilde{n}, n_c$  and  $\tilde{\mu}$ . Thus our system of hydrodynamic equations is closed and can be solved since it involves 11 local variables and 11 equations.

For reasons discussed earlier, the explicit effect of the  $C_{22}$  collisions has disappeared in the hydrodynamic equations (98). However, the  $C_{22}$  collisions justify and enforce the local equilibrium form (94) used for  $f(\mathbf{r}, \mathbf{p}, t)$ , which describes the excited atoms. In contrast,  $C_{12}$  is still explicitly present in (98) through the  $\Gamma_{12}$  source term. This means that the condensate is not in *diffusive* equilibrium with non-condensate. We recall from (95) that

$$(101) \quad \Gamma_{12} \propto g^2 n_c [1 - e^{-\beta(\tilde{\mu}-\mu_c)}],$$

and thus  $\Gamma_{12}[f_0]$  vanishes only when  $\tilde{\mu} = \mu_c$ . One might expect that this condition would arise in the case of strong  $C_{12}$  collisions. A detailed study [44] shows that even in this case, one must allow for fluctuations in the variable  $\mu_{diff} \equiv \tilde{\mu} - \mu_c$  and thus we cannot simply set  $\Gamma_{12} = 0$ . Since  $\Gamma_{12} \neq 0$  in (98), we have to generalize the HF Popov description to ensure that [compare with the result in (81)]

$$(102) \quad \frac{\partial n_c}{\partial t} + \nabla \cdot (n_c \mathbf{v}_s) = -\Gamma_{12}.$$

This is needed for the continuity equation to be correct

$$(103) \quad \frac{\partial n_c}{\partial t} + \nabla \cdot \mathbf{j} = 0,$$

where

$$(104) \quad \begin{aligned} n &= \tilde{n} + n_c \\ j &= \tilde{n} \mathbf{v}_n + n_c \mathbf{v}_s. \end{aligned}$$

As discussed in detail in Ref.[44], a simple microscopic model based on the neglect of anomalous pair correlation functions  $\langle \tilde{\psi}(\mathbf{r}) \tilde{\psi}(\mathbf{r}) \rangle$  leads to both (102) and (98).

We can linearize our coupled two-fluid hydrodynamic equations given by (81) and (98) around the static equilibrium solution, where  $\mu_{c0} = \tilde{\mu}_0$ . The fluctuations  $\delta \tilde{n}, \delta n_c, \delta \mathbf{v}_n, \delta \mathbf{v}_s$  are precisely the quantities which can be directly measured in trapped gases. The only static equilibrium functions we need are  $\tilde{n}_0, n_{c0}$  and  $\tilde{P}_0$ , which are given by (99),

$$(105) \quad \tilde{P}_0 = \frac{1}{\beta_0 \Lambda_0^3} g_{5/2}(z_0), \quad \tilde{n}_0 = \frac{1}{\Lambda_0^3} g_{3/2}(z_0),$$

where  $z_0 = e^{-\beta g n_{c0}(\mathbf{r})}$  and  $n_{c0}$  is given by the *static* HFP equation (52) for  $\Phi(\mathbf{r})$ . We note that these static thermodynamic properties for a Bose-condensed gas are quite different from the expressions first given by Lee and Yang (1958) for a uniform gas [47]. They calculated only the first-order interaction corrections to the thermodynamic properties of an ideal Bose gas, using

$$(106) \quad \begin{aligned} \tilde{P}_0 &\simeq \frac{1}{\beta_0 \Lambda_0^3} g_{5/2}(z=1) - g n_{c0} \tilde{n}_{cr} \\ \tilde{n}_0 &\simeq \frac{1}{\Lambda_0^3} g_{3/2}(z=1) \equiv \tilde{n}_{cr}. \end{aligned}$$

Here  $\tilde{n}_{cr}$  is the critical density at the temperature  $T$  for an ideal uniform Bose gas. The non-perturbative approximation we use for thermodynamic properties [as illustrated by (105)] is consistent with the Lee-Yang results when we expand to first order in  $g$  [48].

We can use our generalized ZGN equations to derive two-fluid equations in the traditional form first given by Landau (1941). In linearized form, these are [6, 49]

$$(107) \quad \begin{aligned} \frac{\partial \delta n}{\partial t} + \nabla \cdot \delta \mathbf{j} &= 0 \\ m \frac{\partial \delta \mathbf{j}}{\partial t} &= -\nabla \delta P - \delta n \nabla U_{\text{ext}}(\mathbf{r}) \\ m \frac{\partial \delta \mathbf{v}_s}{\partial t} &= -\nabla \delta \mu_c \\ \frac{\partial \delta s}{\partial t} &= -\nabla \cdot (s \delta \mathbf{v}_n), \end{aligned}$$



where we have included an external potential,  $s(\mathbf{r}, t)$  is the local entropy density and

$$(108) \quad \begin{aligned} \delta \mathbf{j} &= n_{c0} \delta \mathbf{v}_s + \tilde{n}_0 \delta \mathbf{v}_n \\ \delta n &= \delta n_c + \delta \tilde{n}. \end{aligned}$$

Of course, the Landau equations describe the case when the superfluid and normal fluid are in dynamic local equilibrium with each other. The Landau form of two-fluid equations is very natural for superfluid  $^4\text{He}$ , for which they were developed. They involve the usual thermodynamic variables: pressure, entropy and temperature fluctuations. However, in a trapped gas, the more natural variables are those used in a ZGN-type formulation [7, 44], namely  $\tilde{n}, n_c, \mathbf{v}_n, \mathbf{v}_s$  and  $\mu_{diff}$ .

In the generalized ZGN equations discussed in Ref. [44], the fluctuations around static equilibrium involve fluctuations in  $\delta\Gamma_{12}$ , namely [see (101)]

$$(109) \quad \delta\Gamma_{12} \propto g^2 n_{c0} \delta\mu_{diff}.$$

This leads to an additional equation of motion for  $\delta\mu_{diff}$ , with a characteristic relaxation time  $\tau_\mu$ . This gives rise to a new relaxational mode of the condensate. The ZGN limit [7] corresponds to  $\omega\tau_\mu \gg 1$  while the Landau two-fluid limit corresponds to  $\omega\tau_\mu \ll 1$ . While the relaxational mode is not included, the ZGN equations appear to give a reasonable first approximation for the other hydrodynamic modes, as shown in the explicit results given in Ref.[48]. We note that Landau two-fluid equations given by (107) have been recently used by Shenoy and Ho [50] to work out the hydrodynamic normal modes of a trapped superfluid gas.

The Landau two-fluid equations, which assume total local equilibrium, are valid at all temperatures. Our present analysis is restricted to finite  $T \sim T_{\text{BEC}}$ , but could be generalized to the case of very low temperatures. At  $T = 0$ , of course, the distinction between the collisionless and hydrodynamic region is not a useful one (see, moreover, the comment at the end of Section 7). However, at  $T = 0$ , we have  $\rho_n = 0$  and  $\rho_s = \rho$  and then the Landau two-fluid equations formally reduce to hydrodynamic equations for *one* component:

$$(110) \quad \begin{aligned} \frac{\partial n}{\partial t} + \nabla \cdot n \mathbf{v} &= 0 \\ m \frac{\partial \mathbf{v}}{\partial t} &= -\nabla \mu_c. \end{aligned}$$

These equations are the basis of recent work by Pitaevskii and Stringari [51]. In conjunction with the local density approximation [1]

$$(111) \quad \mu_c(\mathbf{r}, t) = \mu_0[n(\mathbf{r}, t)] + U_{\text{ext}}(\mathbf{r}),$$

where  $\mu_0$  is the chemical potential of a *uniform* interacting Bose-condensed gas at  $T = 0$ , such as given by (62), one may use (110) to find collective mode frequencies past the MFA (or GP approximation).

### 11. – In-phase oscillation of the condensate and non-condensate

Two interesting collective modes exhibited by the ZGN equations involve *rigid* centre-of-mass oscillations of the equilibrium density profiles:

$$(112) \quad \begin{aligned} \tilde{n}(\mathbf{r}, t) &= \tilde{n}_0(\mathbf{r} - \boldsymbol{\eta}_n(t)), & \dot{\boldsymbol{\eta}}_n(t) &= \mathbf{v}_n(t) \\ n_c(\mathbf{r}, t) &= n_{c0}(\mathbf{r} - \boldsymbol{\eta}_c(t)), & \dot{\boldsymbol{\eta}}_c(t) &= \mathbf{v}_s(t). \end{aligned}$$

One finds [7] an in-phase solution (Kohn mode) with  $\mathbf{v}_n = \mathbf{v}_s$ , with a frequency given by the trap frequency  $\omega_\alpha$ ; and an out-of-phase solution, with  $\mathbf{v}_n$  and  $\mathbf{v}_s$  in opposite directions. The latter is the *analogue* of the second sound mode in liquid  $^4\text{He}$ .

First let us work out the static equilibrium solution  $f_{eq}(\mathbf{r}, \mathbf{p})$  of (97), which must satisfy

$$(113) \quad \frac{\mathbf{p}}{m} \cdot \nabla_r f_{eq}(\mathbf{r}, \mathbf{p}) - \nabla U_0(\mathbf{r}) \cdot \nabla_p f_{eq}(\mathbf{r}, \mathbf{p}) = C[f_{eq}] = 0.$$

The solution is easily verified to be

$$(114) \quad f_0(\mathbf{r}, \mathbf{p}) = \frac{1}{e^{\beta_0(\frac{p^2}{2m} + U_0(\mathbf{r}) - \tilde{\mu}_0)} - 1},$$

where (within the Thomas-Fermi approximation for  $\mu_{c0}$ )

$$(115) \quad \begin{aligned} \tilde{\mu}_0 &= \mu_{c0} = U_{\text{ext}}(\mathbf{r}) + 2g\tilde{n}_0(\mathbf{r}) + gn_{c0}(\mathbf{r}) \\ U_0(\mathbf{r}) &= U_{\text{ext}}(\mathbf{r}) + 2g[\tilde{n}_0(\mathbf{r}) + n_{c0}(\mathbf{r})]. \end{aligned}$$

Using (114), we find  $\tilde{n}_0(\mathbf{r})$  is given by (105), with

$$(116) \quad \Lambda_0 = \left( \frac{2\pi}{mk_B T_0} \right)^{1/2}; \quad z_0 \equiv e^{\beta_0(\tilde{\mu}_0 - U_0)} = e^{-\beta_0 g n_{c0}}.$$

We next consider the in-phase solution described by (112), with  $\mathbf{v}_s = \mathbf{v}_n = \dot{\boldsymbol{\eta}}$  (not dependent on the position  $\mathbf{r}$ ). This is a solution of ZGN equations with  $\eta_\alpha(t) \sim e^{i\omega_\alpha t}$ , where  $\omega_\alpha$  is the parabolic trap frequency in  $\alpha^{th}$  direction ( $x, y, z$ ). The proof proceeds as follows (in the linearized theory). Given (112), we have

$$(117) \quad \begin{aligned} \delta n_c(\mathbf{r}, t) &= -\boldsymbol{\eta}(t) \cdot \nabla n_{c0}(\mathbf{r}) \\ \delta \tilde{n}(\mathbf{r}, t) &= -\boldsymbol{\eta}(t) \cdot \nabla \tilde{n}_0(\mathbf{r}), \end{aligned}$$

and hence

$$(118) \quad \begin{aligned} m \frac{\partial \delta \mathbf{v}_s}{\partial t} &= -\nabla \delta \mu_c = -\nabla [2g \delta \tilde{n} + g \delta n_c] \\ &= \nabla (\boldsymbol{\eta} \cdot [2g \nabla \tilde{n}_0 + g \nabla n_{c0}]). \end{aligned}$$

This result is equivalent to [using (115)]

$$(119) \quad m \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} = -\boldsymbol{\nabla} (\boldsymbol{\eta} \cdot \boldsymbol{\nabla} U_{\text{ext}}),$$

with

$$(120) \quad U_{\text{ext}} = \frac{1}{2} m (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2).$$

The solution of (119) is clearly

$$(121) \quad \ddot{\eta}_\alpha = -\omega_\alpha^2 \eta_\alpha, \quad \eta_\alpha(t) \sim e^{i\omega_\alpha t}.$$

In a similar way, one can show that the hydrodynamic equations for the non-condensate lead to the same equation of motion (119). The rigid motion of the non-condensate density profile corresponds to a distribution function given by

$$(122) \quad f(\mathbf{r}, \mathbf{p}, t) = f_0(\mathbf{r} - \boldsymbol{\eta}, \mathbf{p} - m\dot{\boldsymbol{\eta}}).$$

This can be verified to solve the kinetic equation (97), using the fact that  $C[f_0] = 0$ , and leads to the expression in (112), namely

$$(123) \quad \tilde{n}(\mathbf{r}, t) = \int \frac{d\mathbf{p}}{(2\pi)^3} f(\mathbf{r}, \mathbf{p}, t) = \tilde{n}_0(\mathbf{r} - \boldsymbol{\eta}).$$

This in-phase dipole mode was first exhibited by ZGN in the hydrodynamic region [7]. It is an important test of the proper treatment of the non-condensate dynamics. This kind of mode is generic, existing in the collisionless region as well [29, 34].

## 12. – First and second sound in a uniform Bose-condensed gas

The simplicity of the ZGN two-fluid equations allow a very transparent discussion of first and second sound in a uniform gas [48]. We find two sound-wave solutions:  $\omega = u_1 q$ ,  $\omega = u_2 q$ . First sound has a velocity given by (to lowest order in  $g$ )

$$(124) \quad u_1^2 = \frac{5}{3} \frac{\tilde{P}_0}{m\tilde{n}_0} + \frac{2g\tilde{n}_0}{m},$$

and mainly involves the non-condensate, with  $v_n \gg v_s$  (but in-phase). Second sound has a velocity given by (to lowest order in  $g$ )

$$(125) \quad u_2^2 = \frac{gn_{c0}}{m},$$

and mainly involves the oscillation of the condensate, with  $v_s \gg v_n$  (and out-of-phase). It is clear that second sound is the collision-dominated hydrodynamic mode which is the

*analogue* of the collisionless Bogoliubov phonon (42) which we discussed in Section 4. The frequency vanishes (becomes “soft”) above  $T_{\text{BEC}}$  and it is the expected Goldstone mode of the Bose broken-symmetry [16, 17]. In contrast to liquid  $^4\text{He}$ , both second and first sound in a Bose gas involve density fluctuations (ie, second sound is not a temperature wave in a gas!).

Can we study first and second sound pulse propagation in cigar-shaped traps? In this volume, Ketterlee [31] reviews the beautiful pulse propagation experiments carried out at MIT at low temperatures in the collisionless region [52]. If we could get into the hydrodynamic region, one should be able to see *two* pulses propagate—roughly with the speeds  $u_1$  and  $u_2$  given above. This would be very dramatic and a direct confirmation of the superfluid dynamics described by the two-fluid equations. Inside the broad, relatively uniform condensate distribution along the axis of the cigar-shaped trap, one could excite both first and second sound pulses, with the relative intensity of each propagating pulse given by [53]

$$(126) \quad \frac{1}{u_i^2} \delta(z - u_i t).$$

The velocity of second sound would give [using (125)] a direct measurement of the condensate density  $n_{c0}$ . Perhaps of more interest, the velocity of the first sound pulse would give a relatively *direct* measurement [using (124)] of the non-condensate density  $\tilde{n}_0$  “underneath” the condensate.

### 13. – Concluding remarks

In Sections 9 and 10, we discussed the linearized form of the generalized two-fluid equations [44] which describe how the condensate relaxes to equilibrium with the excited atoms. These hydrodynamic equations with  $\Gamma_{12} \neq 0$  (i.e.  $\tilde{\mu} \neq \mu_c$ ) should also be useful in the study of how a condensate grows and comes into equilibrium with the thermal cloud [31]. While based on a different formalism, our theory has points of contact with the work by Gardiner, Zoller and co-workers on the kinetics of condensate growth [45, 46]. However, our analysis, while assuming the existence of a condensate, takes the dynamical evolution of the non-condensate fully into account. The two-fluid dynamics of trapped Bose-condensed gases promises to be a very rich subject, as discussed in Ref. [44]. The region where  $\Gamma_{12} \neq 0$  is not easily studied in superfluid  $^4\text{He}$ , largely because local thermal equilibrium is almost too easy to reach in a liquid. However, some years ago Pitaevskii [54] treated the regime very close to  $T_\lambda$ , when the superfluid density  $\rho_s(\mathbf{r}, t)$  was not necessarily in equilibrium with the normal fluid density  $\rho_n(\mathbf{r}, t)$ .

One of the interesting aspects of trapped Bose gases is that even if the centre of the trap is at large enough density so that one is in the hydrodynamic region described by dynamic local thermal equilibrium, this description will always break down in the low density non-condensate thermal cloud far enough from the trap centre. Recent theoretical studies [55, 56] have shown that the cross-over region between the hydrodynamic and

collisionless regions in this low density region plays the dominant role as the source of damping of certain kinds of hydrodynamic oscillations.

More generally, I would like to emphasize that the study of collective modes in trapped Bose gases finally provides us with the opportunity of making a quantitative test of the complex dynamics of a system with a Bose-broken symmetry. As I discuss elsewhere in this volume [9], the complexity of dealing with a liquid such as superfluid  $^4\text{He}$  never allowed one to make a thorough confrontation of the microscopic theory of Bose-condensed systems with experiment [11]. Bose gases finally allow us to do this, in a much cleaner fashion [31, 57]. I would hope that future studies on trapped Bose gases will also stimulate new interest in the dynamics of superfluid  $^4\text{He}$  as a Bose-condensed liquid.

\* \* \*

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**Figure Captions**

1. Hartree-Fock self-energies in a normal Bose gas.
2. Various interaction terms involving condensate (wiggly line) and non-condensate (solid line) atoms.
3. Self-energy diagrams in the Bogoliubov approximation ( $T = 0$ ).
4. Self-energy diagrams in the first-order Popov approximation. The propagators are for an ideal Bose gas.
5. Self-consistent Hartree-Fock-Bogoliubov (HFB) approximation for self-energies.
6. Ladder diagram approximation for the  $t$ -matrix.